Timing Channels and Shift-Correcting Codes

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Contents

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- C. E. Shannon, "The Zero Error Capacity of a Noisy Channel," *IRE Trans. Inf. Theory* 2 (3), 1956.

Zero-Error Communication

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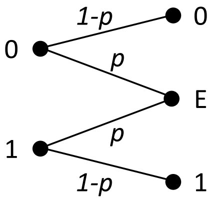

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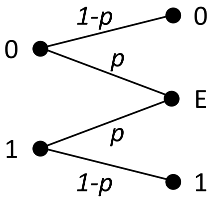

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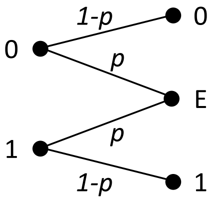

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  - Every two codewords are confusable $\Longrightarrow$ the zero-error capacity of the BEC is equal to zero

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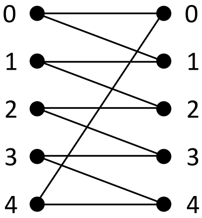

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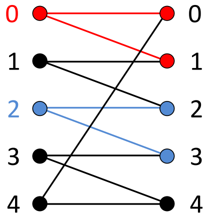

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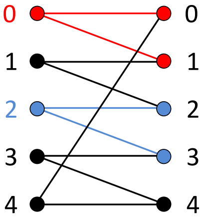

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  - L. Lovasz, "On the Shannon Capacity of a Graph," *IEEE Trans. Inf. Theory* 25 (1), 1979. (IT paper award)

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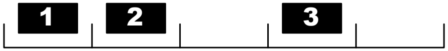

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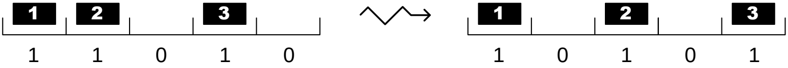

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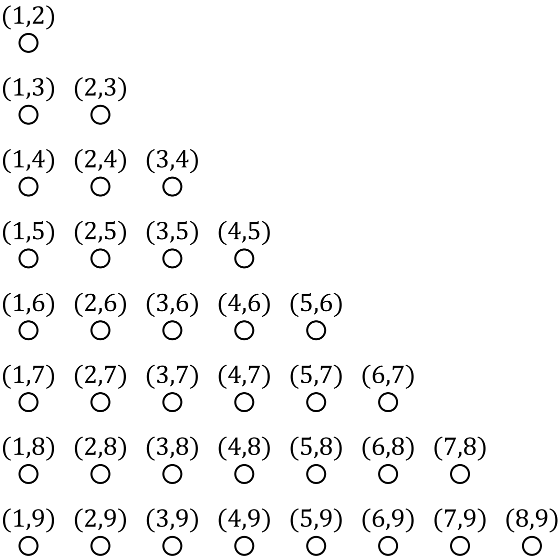
- Example: $n = 9$, $W = 2$, $K = 1$

Optimal zero-error codes: Geometric approach



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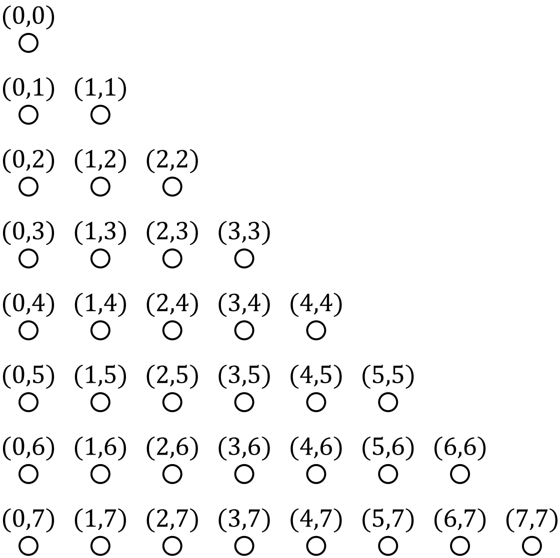

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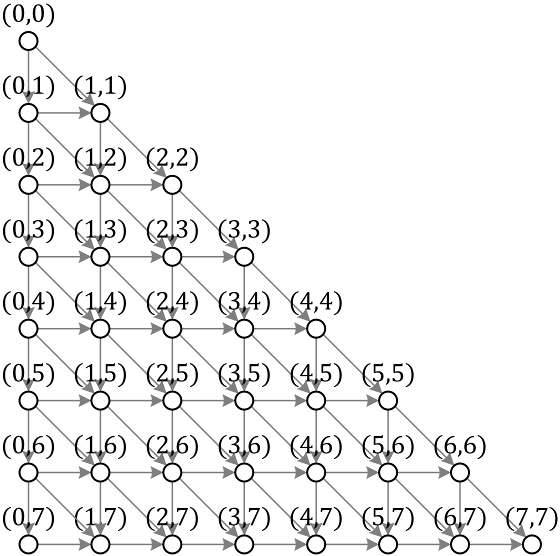

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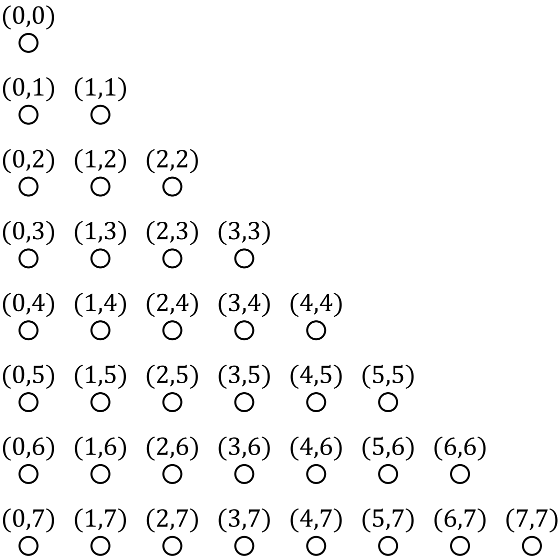

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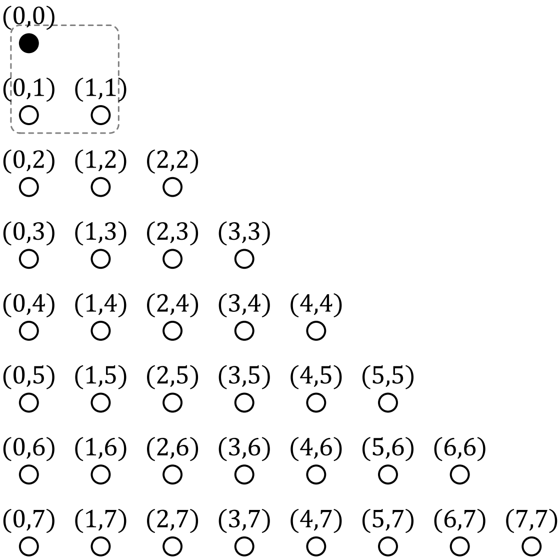

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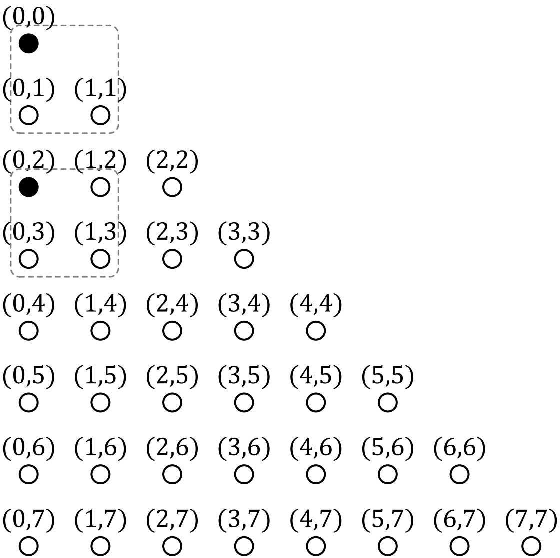

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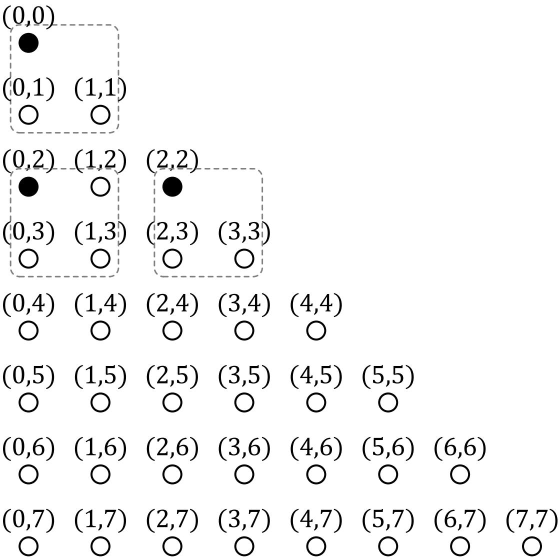

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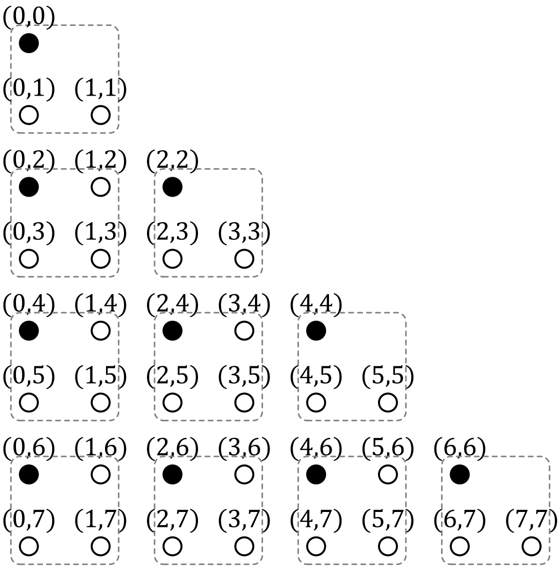

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Theorem

The zero-error capacity of the Shift Channel with parameter K is equal to $\log r$, where r is the unique positive real root of the polynomial $x^{K+1} - x^K - 1$.

Zero-error capacity

- Proof:
  - $M(n)$ can also be described recursively

    $$M(n) = M(n-1) + M(n-K-1)$$

    with $M(n) = n+1$ for $n \leq K$
  - This implies that

    $$M(n) = \sum_{k=0}^{K} a_k r_k^n$$

    where r_k are the roots of the polynomial $x^{K+1} - x^K - 1$, and a_k are complex constants
  - Therefore, $M(n) \sim ar^n$, where r is the largest of these roots (which is the unique positive real root)

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$$C_0(\omega) = \lim_{n\to\infty} \frac{1}{n} \log M(n, \omega n) = \frac{\omega K + 1}{K+1} \mathcal{H}\left(\frac{\omega(K+1)}{\omega K + 1}\right)$$

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- The zero-error capacity can be achieved with constant-weight codes, so

$$C_0 = \max_{\omega\in[0,1]} C_0(\omega) = \frac{\omega^* K + 1}{K+1} \mathcal{H}\left(\frac{\omega^*(K+1)}{\omega^* K + 1}\right)$$

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    - This is quantified by the second-order term $-\frac{1}{2n}\log n$

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- Continuous-time channel with emissions separated by at least τ seconds, and with the maximum delay of T seconds
  - The capacity equals $\frac{1}{\tau} \log r$, where r is the unique positive root of the polynomial $x^{T/\tau} - x^{T/\tau - 1} - 1$

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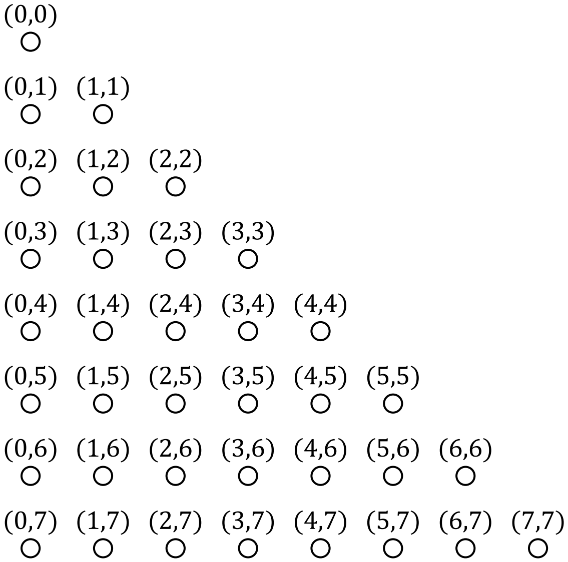
- *Zero-error-detection capacity* of a channel is the largest rate achievable (asymptotically) with zero-error-detecting codes

Zero-error-detecting codes: Example ($K = 2$)



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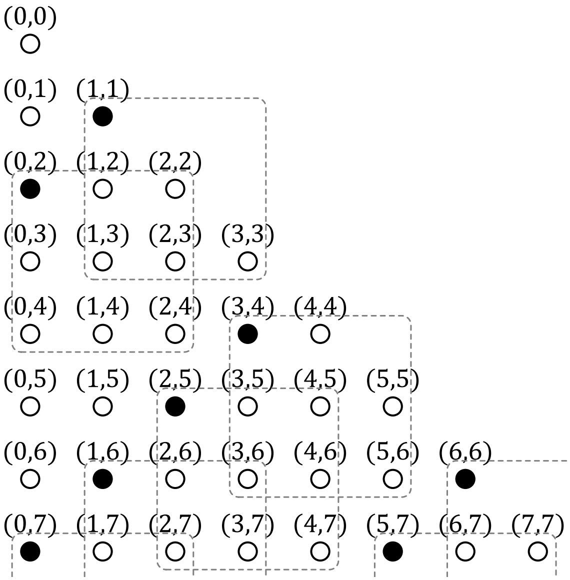

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  - This code is a subcode of $\mathcal{C}(n, W)$ obtained as its intersection with the hyperplanes $\sum_{i=1}^{W} x_i = a \pmod{WK_2 + 1}$

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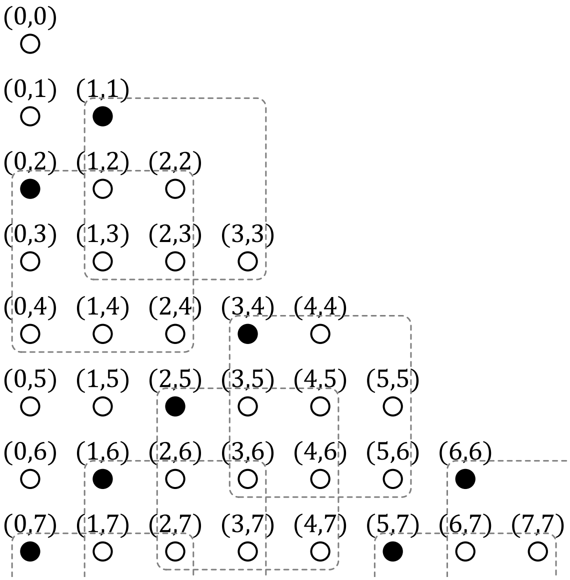
  - This code is a subcode of $\mathcal{C}(n, W)$ obtained as its intersection with the hyperplanes $\sum_{i=1}^{W} x_i = a \pmod{WK_2 + 1}$
  - Example: $n = 9, \; W = 2, \; K_1 = 0, \; K_2 = 2$

Zero-error-detecting codes: Construction $(K_1 = 0, K_2 = 2)$



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- Proof:
  - Let $\mathbf{y}$ be the received sequence
  - If $\sum_{i=1}^{W} y_i \neq a \pmod{WK_2 + 1}$, the receiver detects an error
  - Suppose then that $\sum_{i=1}^{W} y_i = a \pmod{WK_2 + 1}$
  - This means that the sum of the coordinates has been changed in the channel by a multiple of $WK_2 + 1$
  - Since $-K_1 \leq y_i - x_i \leq K_2$ in our model, we have $-WK_1 \leq \sum_{i=1}^{W}(y_i - x_i) \leq WK_2$, so the sum could not have been changed for a nonzero multiple of $WK_2 + 1$
  - Therefore, the sum wasn't changed at all and, if there were any shifts in channel, some of them must have been shifts to the right and some of them to the left
  - Suppose that the i'th particle was shifted to the left, $y_i < x_i$
  - Then, since x_i is a multiple of $K_1 + 1$, and $-K_1 \leq y_i - x_i < 0$, y_i cannot be a multiple of $K_1 + 1$, and so $\mathbf{y}$ is not a codeword

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  - Asymptotically, as $n \to \infty$ and $W \sim \omega n$, the codes $\mathcal{D}^{(a)}(n, W)$ have the same rate as the codes $\mathcal{C}(n, W)$ designed for the smaller of the two parameters K_1, K_2

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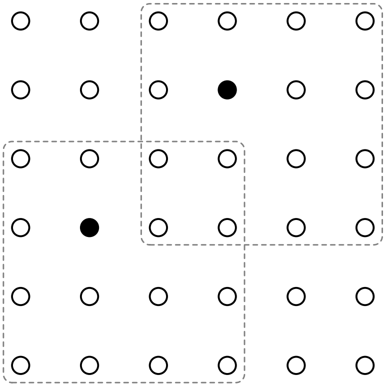

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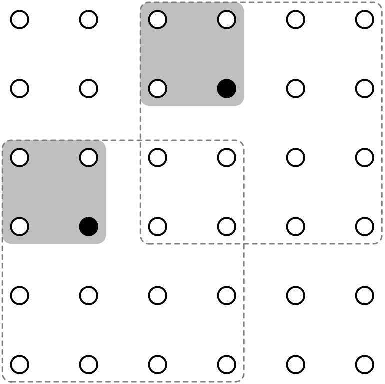

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Theorem

The zero-error-detection capacity of the Shift Channel with parameters K_1, K_2, is equal to $\log r$*, where* r *is the unique positive real root of the polynomial* $x^{\min\{K_1,K_2\}+1} - x^{\min\{K_1,K_2\}} - 1$.

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- ...which is the same as the zero-error-correction capacity of the Shift Channel with parameters 0, $\min\{K_1, K_2\}$

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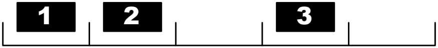

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- The received sequence can be as long as $(K+1)n$ (longer than the input for a multiplicative constant!)
- We have to incorporate this fact in the definition of the code rate:

$$\frac{1}{L_{\text{av}}(n)} \log M(n)$$

where $L_{\text{av}}(n)$ is the average output length (average over all codewords and channel statistics)

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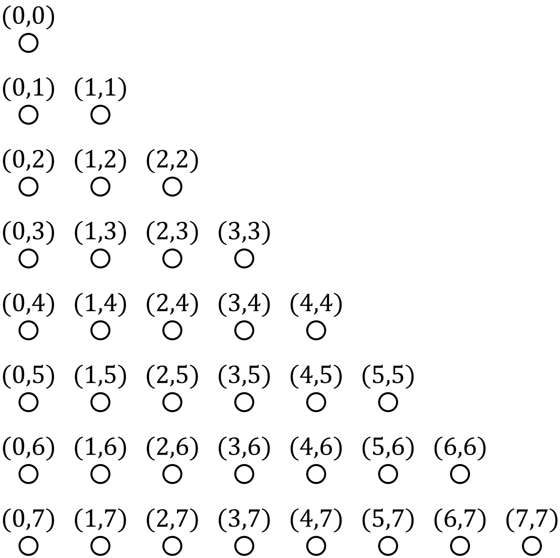

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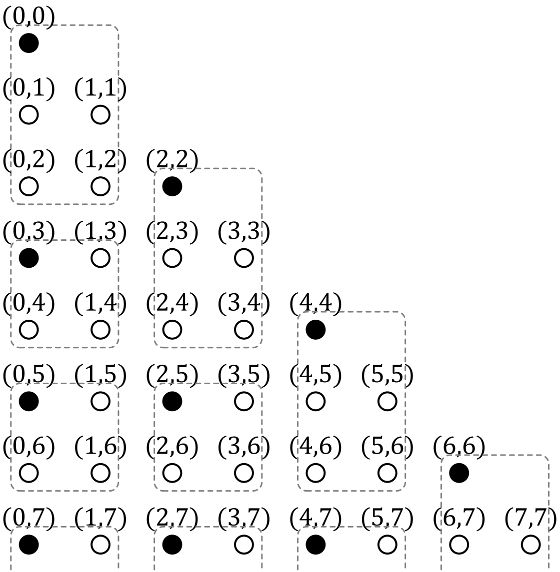

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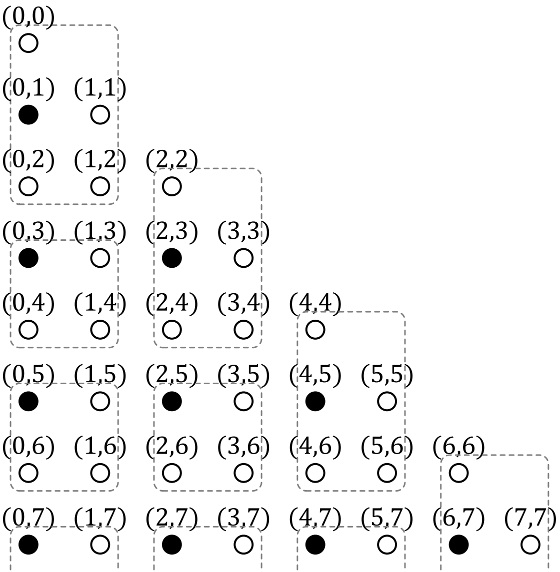

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$$C_0^{\mathrm{Q}} = \frac{1}{K+1}$$

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Zero-error capacity of the DTQP with P types of packets is

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- Even though we are analyzing the zero-error case, the capacity depends on the channel statistics through $\bar{\kappa} = \sum_{k=0}^{K} k \cdot p(k)$

And Finally...

Thank you for your attention!