Schubert Calculus over Finite Fields

COST Training School, Barcelona, February 4-8, 2013.

Joachim Rosenthal
Institut für Mathematik
University of Zürich

Grassmann variety and subspace codes

Let V be a vector space of dimension n over a field $\mathbb{F}$.

Grassmann variety and subspace codes

Let V be a vector space of dimension n over a field $\mathbb{F}$.

Definition

The Grassmann variety $\mathrm{Grass}(k, V)$ is the set of all k-dimensional subspaces $U \subset V$.

Grassmann variety and subspace codes

Let V be a vector space of dimension n over a field $\mathbb{F}$.

Definition

The Grassmann variety $\mathrm{Grass}(k, V)$ is the set of all k-dimensional subspaces $U \subset V$.

Remark

A subset $\mathbf{C} \subset \mathrm{Grass}(k, V)$ can be viewed as a constant dimensional subspace code.

Grassmann variety and subspace codes

Let V be a vector space of dimension n over a field $\mathbb{F}$.

Definition

The Grassmann variety $\mathrm{Grass}(k, V)$ is the set of all k-dimensional subspaces $U \subset V$.

Remark

A subset $\mathbf{C} \subset \mathrm{Grass}(k, V)$ can be viewed as a constant dimensional subspace code.

Question: Why is $\mathrm{Grass}(k, V)$ a variety?

Plücker Embedding

Consider the vector space of alternating k–tensors $\wedge^k V$. Let $\mathbb{P}(\wedge^k V)$ be the projective space consisting of all lines in $\wedge^k V$.

Plücker Embedding

Consider the vector space of alternating k–tensors $\wedge^k V$. Let $\mathbb{P}(\wedge^k V)$ be the projective space consisting of all lines in $\wedge^k V$. The Plücker embedding is defined through:

$$\begin{array}{rrcl} \varphi : & \mathrm{Grass}(k, V) & \longrightarrow & \mathbb{P}(\wedge^k V) \\ & \mathrm{span}(v_1, \ldots, v_k) & \longmapsto & \mathbb{F} v_1 \wedge \cdots \wedge v_k . \end{array} \tag{1}$$

Background from linear algebra

Let $\{e_1, \ldots, e_n\}$ be a basis of V.

Background from linear algebra

Let $\{e_1, \ldots, e_n\}$ be a basis of V.

Then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a basis of $\wedge^k V$.

Background from linear algebra

Let $\{e_1, \ldots, e_n\}$ be a basis of V.

Then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a basis of $\wedge^k V$.

In particular:

$$\dim \wedge^k V = \binom{\dim V}{k} = \binom{n}{k}$$

Background from linear algebra

Let $\{e_1, \ldots, e_n\}$ be a basis of V.

Then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a basis of $\wedge^k V$.

In particular:

$$\dim \wedge^k V = \binom{\dim V}{k} = \binom{n}{k}$$

In particular: $\wedge^n V = \mathbb{F}$ and $\wedge^{n+1} V = \{0\}$.

Background from linear algebra

Let $\{e_1, \ldots, e_n\}$ be a basis of V.

Then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a basis of $\wedge^k V$.

In particular:

$$\dim \wedge^k V = \binom{\dim V}{k} = \binom{n}{k}$$

In particular: $\wedge^n V = \mathbb{F}$ and $\wedge^{n+1} V = \{0\}$.
Note that the expressions $v_1 \wedge \cdots \wedge v_k$ are linear in each component and alternating.

Plücker Coordinates

Assume

$$v_i = \sum_{j=1}^{n} a_{ij} e_j, \; i = 1, \ldots, k.$$

Let A be the $k \times n$ matrix $(a_{i,j})$. The Plücker embedding writes:

$$\begin{array}{rrcl} \varphi \; : & Mat_{k\times n} & \longrightarrow & \mathbb{P}(\wedge^k V) \\ & \mathrm{rowspace}(A) & \longmapsto & \sum_{1\leq i_1 < \cdots < i_k \leq n} x_{i_1,\ldots,i_k} \cdot e_{i_1} \wedge \ldots \wedge e_{i_k}. \end{array} \tag{2}$$

The coordinates $x_{\underline{i}} := x_{i_1,\ldots,i_k}$ are called the Plücker coordinates of $\mathrm{rowspace}(A)$.

Example Plücker Coordinates

Consider the subspace

$$\text{rowspace}\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 6 \end{pmatrix}$$

Example Plücker Coordinates

Consider the subspace

$$\text{rowspace}\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 6 \end{pmatrix}$$

Then its Plücker coordinates are:

$$(1, 3, 6, -2, 0, 12).$$

Example Plücker Coordinates

Consider the subspace

$$\text{rowspace}\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 6 \end{pmatrix}$$

Then its Plücker coordinates are:

$$(1, 3, 6, -2, 0, 12).$$

Question: Does every 6-vector appear as the Plücker coordinates of some subspace?

Example Plücker Coordinates

Consider the subspace

$$\text{rowspace}\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 6 \end{pmatrix}$$

Then its Plücker coordinates are:

$$(1, 3, 6, -2, 0, 12).$$

Question: Does every 6-vector appear as the Plücker coordinates of some subspace?

Answer: No.

Shuffle Relations

Theorem

$$\sum_{\lambda=1}^{k+1} (-1)^{\lambda} \cdot x_{i_1,\ldots,i_{k-1},j_\lambda} \cdot x_{j_1,\ldots,\hat{j_\lambda},\ldots,j_{k+1}} = 0 \tag{3}$$

describes the image of the Grassmannian in the projective space $\mathbb{P}(\wedge^k V)$

Shuffle Relations

Theorem

$$\sum_{\lambda=1}^{k+1}(-1)^{\lambda}\cdot x_{i_1,\ldots,i_{k-1},j_\lambda}\cdot x_{j_1,\ldots,\hat{j_\lambda},\ldots,j_{k+1}}=0 \tag{3}$$

describes the image of the Grassmannian in the projective space $\mathbb{P}(\wedge^k V)$

Example

Grass$(2,\mathbb{F}^4)$ is embedded in $\mathbb{P}^5$ and $\varphi(Grass(2,4))$ is described by a single relation

$$x_{12}x_{34}-x_{13}x_{24}+x_{14}x_{23}=0 \tag{4}$$

Shuffle Relations

Example

Grass$(2, \mathbb{F}^5)$ is embedded in $\mathbb{P}^9$ and the defining relations are:

$$\begin{aligned}
x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} &= 0 \\
x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23} &= 0 \\
x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{14} &= 0 \\
x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34} &= 0 \\
x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} &= 0
\end{aligned}$$

Importance in Network Coding

Metric on Grassmannian: If $U, W \in \mathrm{Grass}(k, V)$ are two subspaces one defines its distance as:

$$d(U,W) := \dim(U+W) - \dim(U \cap W).$$

Importance in Network Coding

Metric on Grassmannian: If $U, W \in \mathrm{Grass}(k, V)$ are two subspaces one defines its distance as:

$$d(U,W) := \dim(U+W) - \dim(U \cap W).$$

Question: What is the algebraic structure of the balls of radius t around an element $W \in \mathrm{Grass}(k, V)$?

Importance in Network Coding

Metric on Grassmannian: If $U, W \in \mathrm{Grass}(k, V)$ are two subspaces one defines its distance as:

$$d(U, W) := \dim(U + W) - \dim(U \cap W).$$

Question: What is the algebraic structure of the balls of radius t around an element $W \in \mathrm{Grass}(k, V)$?

Answer: $d(U, W) \leq t$ if and only if $\dim(U \cap W) \geq k - t/2 =: r$.

Importance in Network Coding

Metric on Grassmannian: If $U, W \in \mathrm{Grass}(k, V)$ are two subspaces one defines its distance as:

$$d(U, W) := \dim(U + W) - \dim(U \cap W).$$

Question: What is the algebraic structure of the balls of radius t around an element $W \in \mathrm{Grass}(k, V)$?

Answer: $d(U, W) \leq t$ if and only if $\dim(U \cap W) \geq k - t/2 =: r$.

Remark

The ball of radius t around the subspace W defines a so called Schubert variety:

$$\{U \in \mathrm{Grass}(k, V) \mid d(U, W) \leq t\}$$

Geometric Questions of Schubert

Hermann Schubert studied in the 19th century geometric questions of the following type:

Geometric Questions of Schubert

Hermann Schubert studied in the 19th century geometric questions of the following type:

Example

Given 4 lines in 3-space in general position. Is there a line intersecting all 4 lines.

Geometric Questions of Schubert

Hermann Schubert studied in the 19th century geometric questions of the following type:

Example

Given 4 lines in 3-space in general position. Is there a line intersecting all 4 lines.

Answer Schubert: By Poncelet's principle of conservation of numbers we can assume lines 1 and 2 intersect and lines 3 and 4 intersect. So there are 2 solutions in general.

Geometric Questions of Schubert

Theorem (Schubert [Sch79])

Given $N := k(n-k)$ *linear subspace* $U_i, i = 1, \ldots, N$ *in* V *having dimension* k *each. If the base field* $\mathbb{F}$ *is algebraically closed and the subspaces are in general position then there exist exactly*

$$\frac{1!2!\cdots(k-1)!(N)!}{(n-k)!(n-k+1)!\cdots(n-1)!} \tag{5}$$

subspaces W *of dimension* $(n-k)$ *intersecting each of the subspaces* U_i *nontrivially.*

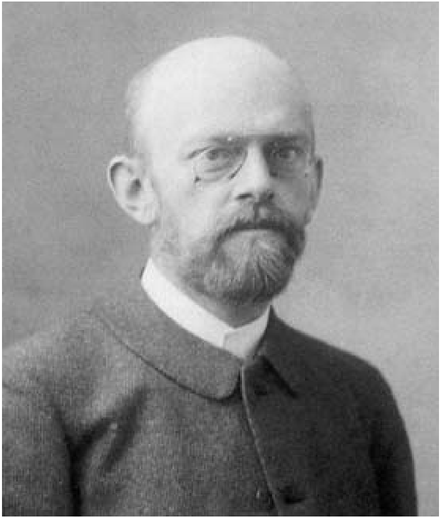
Hermann Cäsar Hannibal Schubert (1848-1911)

Hilbert Problem Number 15, Paris 1900

Rigorous foundation of Schubert's enumerative calculus

The problem consists in this: To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.

Although the algebra of today guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen.



David Hilbert (1862-1943)

Schubert Varieties

Definition

A flag $\mathscr{F}$ is a sequence of nested subspaces

$$\{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_n = V \quad (6)$$

where we assume that $\dim V_i = i$ for $i = 1, \ldots, n$.

Let $\underline{i} = (i_1, \ldots, i_k)$ denote a sequence of numbers having the property that

$$1 \leq i_1 < \ldots < i_k \leq n. \quad (7)$$

Definition

For each flag $\mathscr{F}$ and each multiindex $\underline{i}$

$$C(\underline{i}; \mathscr{F}) := \{W \in \mathrm{Grass}(k, V) \mid \dim(W \bigcap V_{i_s}) = s\}$$

is called a Schubert cell.

Schubert Varieties

Definition

For each flag $\mathscr{F}$ and each multiindex $\underline{i}$

$$S(\underline{i};\mathscr{F}) := \{W \in \mathrm{Grass}(k,V) \mid \dim(W \bigcap V_{i_s}) \geq s\}$$

is called a Schubert variety.

Schubert Varieties

Definition

For each flag $\mathscr{F}$ and each multiindex $\underline{i}$

$$S(\underline{i};\mathscr{F}) := \{W \in \text{Grass}(k,V) \mid \dim(W \bigcap V_{i_s}) \geq s\}$$

is called a Schubert variety.

Remark

The closure of the cell $C(\underline{i};\mathscr{F})$ is the Schubert variety $S(\underline{i};\mathscr{F})$. The equations describing the variety $S(\underline{i};\mathscr{F})$ consists of the quadratic equations describing the Grassmann variety and some additional linear equations.

Schubert Varieties

Remark

If $\{e_1, \dots, e_n\}$ is a basis of V and $\mathscr{F}$ is the standard flag with respect to this basis then $C(\underline{i}; \mathscr{F})$ consists of all subspaces having a certain row reduced echelon form:

$$\begin{bmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * & 0 & \cdots & * & \cdots & * & 1 & 0 & \cdots & 0 \end{bmatrix}$$

Central Question of Schubert Calculus

Problem

Given two Schubert varieties $S(v;\mathscr{F})$ and $S(\tilde{v};\tilde{\mathscr{F}})$. Describe as explicitly as possible the intersection variety

$$S(v;\mathscr{F}) \cap S(\tilde{v};\tilde{\mathscr{F}}).$$

Central Question of Schubert Calculus

Problem

Given two Schubert varieties $S(v;\mathscr{F})$ and $S(\tilde{v};\tilde{\mathscr{F}})$. Describe as explicitly as possible the intersection variety

$$S(v;\mathscr{F})\cap S(\tilde{v};\tilde{\mathscr{F}}).$$

Remark

Schubert's Theorem can actually also be formulated as an intersection problem of Schubert varieties. For this note that

$$\left\{\mathscr{V}\in \mathrm{Grass}(k,\mathbb{F}^{k+m}) \mid \mathscr{V}\bigcap \mathscr{U}_i\neq\{0\}\right\} \tag{8}$$

is a Schubert variety.

Defining Equations of Schubert Varieties

Bruhat order:
Let $\underline{i} := (i_1, \ldots, i_k)$ and $\underline{j} := (j_1, \ldots, j_k)$ be two set of indices satisfying

$$1 \leq i_1 < \ldots < i_k \leq n$$

respectively

$$1 \leq j_1 < \ldots < j_k \leq n.$$

Then one defines:

$$\underline{i} \leq \underline{j}$$

if and only if $i_t \leq j_t$ for $t = 1, \ldots, k$.

Defining Equations of Schubert Varieties

Bruhat order:
Let $\underline{i} := (i_1, \ldots, i_k)$ and $\underline{j} := (j_1, \ldots, j_k)$ be two set of indices satisfying

$$1 \leq i_1 < \ldots < i_k \leq n$$

respectively

$$1 \leq j_1 < \ldots < j_k \leq n.$$

Then one defines:

$$\underline{i} \leq \underline{j}$$

if and only if $i_t \leq j_t$ for $t = 1, \ldots, k$.

Theorem

The defining equations in terms of Plücker coordinates of the Schubert variety $S(\underline{i}; \mathscr{F})$ are given by the quadratic shuffle relations together with the linear equations $x_{\underline{j}} = 0$ for all $\underline{j} \not\leq \underline{i}$.

Application to Sum of Hermitian Matrices

Given Hermitian matrices $A_1, \ldots, A_r \in \mathbb{C}^{n \times n}$ each with a fixed spectrum

$$\lambda_1(A_l) \geq \ldots \geq \lambda_n(A_l), \quad l = 1, \ldots, r \tag{9}$$

and arbitrary else.

Application to Sum of Hermitian Matrices

Given Hermitian matrices $A_1, \ldots, A_r \in \mathbb{C}^{n\times n}$ each with a fixed spectrum

$$\lambda_1(A_l) \geq \ldots \geq \lambda_n(A_l), \quad l = 1, \ldots, r \tag{9}$$

and arbitrary else.
Using the ordered set of eigenvectors one constructs for each Hermitian matrix A_l the flag:

$$\mathscr{F}_l : \quad \{0\} \subset V_{1l} \subset V_{2l} \subset \ldots \subset V_{nl} = \mathbb{C}^n \tag{10}$$

defined through the property:

$$V_{ml} := \mathrm{span}(v_{1l}, \ldots, v_{ml}) \;\; m = 1, \ldots, n. \tag{11}$$

Result of Helmke and Rosenthal [HR95]

Let $A_1, \dots, A_r$ be complex Hermitian $n \times n$ matrices with associated flags $\mathscr{F}_1, \dots, \mathscr{F}_{r+1}$. Assume $A_{r+1} = A_1 + \cdots + A_r$ and let $\underline{i}_l = (i_{1l}, \dots, i_{kl})$ be $r+1$ sequences of integers satisfying

$$1 \leq i_{1l} < \ldots < i_{kl} \leq n, \quad l = 1, \dots, r+1. \tag{12}$$

If the $r+1$ Schubert subvarieties satisfy:

$$S(\underline{i}_1; \mathscr{F}_1) \bigcap \ldots \bigcap S(\underline{i}_{r+1}; \mathscr{F}_{r+1}) \neq \emptyset. \tag{13}$$

Result of Helmke and Rosenthal [HR95]

Let $A_1, \ldots, A_r$ be complex Hermitian $n \times n$ matrices with associated flags $\mathscr{F}_1, \ldots, \mathscr{F}_{r+1}$. Assume $A_{r+1} = A_1 + \cdots + A_r$ and let $\underline{i}_l = (i_{1l}, \ldots, i_{kl})$ be $r+1$ sequences of integers satisfying

$$1 \leq i_{1l} < \ldots < i_{kl} \leq n, \quad l = 1, \ldots, r+1. \tag{12}$$

If the $r+1$ Schubert subvarieties satisfy:

$$S(\underline{i}_1; \mathscr{F}_1) \bigcap \ldots \bigcap S(\underline{i}_{r+1}; \mathscr{F}_{r+1}) \neq \emptyset. \tag{13}$$

Then the following matrix eigenvalue inequalities hold:

$$\sum_{j=1}^{k} \lambda_{n-i_{j,r+1}+1}(A_1 + \cdots + A_r) \geq \sum_{l=1}^{r} \sum_{j=1}^{k} \lambda_{i_{jl}}(A_l) \tag{14}$$

$$\sum_{j=1}^{k} \lambda_{i_{j,r+1}}(A_1 + \cdots + A_r) \leq \sum_{l=1}^{r} \sum_{j=1}^{k} \lambda_{n-i_{jl}+1}(A_l). \tag{15}$$

Symbolic Schubert Calculus

Theorem

For every fixed flag $\mathscr{F}$ the Schubert cells $C(\underline{i};\mathscr{F})$ decompose the Grassmann variety $\mathrm{Grass}(k,\mathbb{C}^n)$ into a finite cellular CW–complex. The integral homology $H_{2m}(\mathrm{Grass}(k,\mathbb{C}^n),\mathbb{Z})$ has no torsion and is freely generated by the fundamental classes of the Schubert varieties $S(\underline{i};\mathscr{F})$ of real dimension $2m$.

Symbolic Schubert Calculus

Theorem

For every fixed flag $\mathscr{F}$ the Schubert cells $C(\underline{i};\mathscr{F})$ decompose the Grassmann variety $\mathrm{Grass}(k,\mathbb{C}^n)$ into a finite cellular CW–complex. The integral homology $H_{2m}(\mathrm{Grass}(k,\mathbb{C}^n),\mathbb{Z})$ has no torsion and is freely generated by the fundamental classes of the Schubert varieties $S(\underline{i};\mathscr{F})$ of real dimension $2m$.

The Poincaré-dual of the class $(i_1,\ldots,i_k)$ will be denoted by

$$\{\mu_1,\ldots,\mu_k\} := \{n-k-i_1+1, n-k-i_2+2,\ldots,n-i_k\}. \qquad (16)$$

viewed as an element of the cohomolgy ring $H^*(\mathrm{Grass}(k,\mathbb{C}^n),\mathbb{Z})$.

Schubert Calculus

The cohomology ring

$$H^*(\mathrm{Grass}(k, \mathbb{C}^n), \mathbb{Z}) := \bigoplus_{m=0}^{k(n-k)} H^{2m}(k, \mathbb{C}^n), \mathbb{Z}) \tag{17}$$

has in a natural way the structure of a graded ring.

Schubert Calculus

The cohomology ring

$$H^*(\mathrm{Grass}(k, \mathbb{C}^n), \mathbb{Z}) := \bigoplus_{m=0}^{k(n-k)} H^{2m}(k, \mathbb{C}^n), \mathbb{Z}) \qquad (17)$$

has in a natural way the structure of a graded ring.

$$\sigma_j := \{j, 0, \ldots, \ldots, 0\} \quad j = 1, \ldots, n-k. \qquad (18)$$

denotes the jth Chern class.

Schubert Calculus

Computations in the cohomology ring are done by the classical formulas of Pieri and Giambelli and by the Littlewood Richardson rule:

Schubert Calculus

Computations in the cohomology ring are done by the classical formulas of Pieri and Giambelli and by the Littlewood Richardson rule:

Pieri's formula:

$$\{\mu_1, \ldots, \mu_k\} \cdot \sigma_j = \sum_{\substack{\mu_{i-1} \geq \nu_i \geq \mu_i \\ \sum_{i=1}^{k} \nu_i = (\sum_{i=1}^{k} \mu_i) + j}} \{\nu_1, \ldots, \nu_k\}$$

Schubert Calculus

Computations in the cohomology ring are done by the classical formulas of Pieri and Giambelli and by the Littlewood Richardson rule:

Pieri's formula:

$$\{\mu_1, \ldots, \mu_k\} \cdot \sigma_j = \sum_{\substack{\mu_{i-1} \geq \nu_i \geq \mu_i \\ \sum_{i=1}^k \nu_i = (\sum_{i=1}^k \mu_i) + j}} \{\nu_1, \ldots, \nu_k\}$$

Giambelli's formula:

$$\{\mu_1, \ldots, \mu_k\} = = \det \begin{pmatrix} \sigma_{\mu_1} & \sigma_{\mu_1+1} & \cdots & \sigma_{\mu_1+k-1} \\ \sigma_{\mu_2-1} & \sigma_{\mu_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{\mu_k-k+1} & & \cdots & \sigma_{\mu_k} \end{pmatrix}$$

Example of Schubert Calculus

Example

Given 4 lines in 3-space in general position. Is there a line intersecting all 4 lines.

Example of Schubert Calculus

Example

Given 4 lines in 3-space in general position. Is there a line intersecting all 4 lines.

Geometric Problem: Intersection of Schubert varieties of the form $S(2,4)$ inside the Grassmannian $\mathrm{Grass}(2,\mathbb{F}^4)$.

Example of Schubert Calculus

Example

Given 4 lines in 3-space in general position. Is there a line intersecting all 4 lines.

Geometric Problem: Intersection of Schubert varieties of the form $S(2,4)$ inside the Grassmannian $\mathrm{Grass}(2,\mathbb{F}^4)$.

Algebraic Problem: One has the equation of $\mathrm{Grass}(2,\mathbb{F}^4)$:

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$$

together with 4 linear equations describing the 4 Schubert varieties. $\longmapsto$ 2 Solutions.

Example of Schubert Calculus

Example

Given 4 lines in 3-space in general position. Is there a line intersecting all 4 lines.

Geometric Problem: Intersection of Schubert varieties of the form $S(2,4)$ inside the Grassmannian $\mathrm{Grass}(2,\mathbb{F}^4)$.

Algebraic Problem: One has the equation of $\mathrm{Grass}(2,\mathbb{F}^4)$:

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$$

together with 4 linear equations describing the 4 Schubert varieties. $\longmapsto$ 2 Solutions.

Cohomology ring:

$$\{1,0\}\{1,0\}\{1,0\}\{1,0\} = 2\{2,2\}.$$

List Decoding Problem

Given a subspace code $\mathscr{C} \subset \mathrm{Grass}(k, V)$ and a received subspace $W \subset V$, whose dimension is not necessarily k.

List Decoding Problem

Given a subspace code $\mathscr{C} \subset \mathrm{Grass}(k, V)$ and a received subspace $W \subset V$, whose dimension is not necessarily k.

Consider the Schubert variety.

$$S_W := \{U \in \mathrm{Grass}(k, V) \mid d(U, W) \leq t\}$$

List Decoding Problem

Given a subspace code $\mathscr{C} \subset \mathrm{Grass}(k, V)$ and a received subspace $W \subset V$, whose dimension is not necessarily k.

Consider the Schubert variety.

$$S_W := \{U \in \mathrm{Grass}(k, V) \mid d(U, W) \leq t\}$$

Find efficient methods to compute:

$$S_W \bigcap \mathscr{C}$$

List Decoding Problem

Given a subspace code $\mathscr{C} \subset \mathrm{Grass}(k, V)$ and a received subspace $W \subset V$, whose dimension is not necessarily k.

Consider the Schubert variety.

$$S_W := \{U \in \mathrm{Grass}(k, V) \mid d(U, W) \leq t\}$$

Find efficient methods to compute:

$$S_W \bigcap \mathscr{C}$$

- W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer Verlag, Berlin, Heidelberg, New York, 1984.
- H. Hiller, *Combinatorics and intersections of Schubert varieties*, Comment. Math. Helv. **57** (1982), no. 1, 41–59.
- U. Helmke and J. Rosenthal, *Eigenvalue inequalities and Schubert calculus*, Mathematische Nachrichten **171** (1995), 207 – 225.
- B. Huber, F. Sottile, and B. Sturmfels, *Numerical Schubert calculus*, J. Symbolic Comput. **26** (1998), no. 6, 767–788, Symbolic numeric algebra for polynomials.
- S. L. Kleiman, *Problem 15: Rigorous foundations of Schubert's enumerative calculus*, Proceedings of Symposia in Pure Mathematics, vol. 28, Am. Math. Soc., 1976, pp. 445–482.

H. Schubert, *Kalkühl der abzählenden geometrie*, Teubner, Leipzig, 1879.

______, *Anzahlbestimmung für lineare Räume beliebiger Dimension*, Acta Math. **8** (1886), 97–118.